Electrostatic beam modes in a free-electron laser with a coaxial wiggler

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An analysis of the propagation of an electrostatic beam space-charge wave through a coaxial wiggler magnetic field and an axial-guide magnetic field is presented. Equations for the electron orbital velocity, rate of change of axial velocity with Lorentz factor, and the dispersion relation are derived. The effects of the combined wiggler and axial-guide fields on the rate of change of the axial velocity with electron energy and on the effective electron density in the dispersion relation are studied numerically. The present theory predicts that electrostatic negative mass instabilities do not exist in a coaxial wiggler. [S1063-651X(98)03802-1]

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I. INTRODUCTION

The electrostatic stability of a free-electron laser in which a relativistic electron beam passes through combined wiggler and axial guide fields has previously been investigated [1–3]. It has been shown that electrostatic beam space-charge waves can be considerably modified by the combined helical or planar wiggler and axial-guide magnetic fields. Furthermore, instability has been shown to exist in these fields for a certain specific parameter regime. The analysis has been extended recently to include the effects of a cylindrical metallic waveguide wall [4]. The combined effects of the wall, wiggler, and guide field on cyclotronlike waves were studied in detail.

Some studies of the feasibility of using coaxial wigglers in free-electron lasers have been done recently. Freund *et al.* [5,6] analyzed the performance of a coaxial hybrid iron wiggler. The essential parts consist of a central rod and a coaxial ring of alternating ferrite and dielectric spacers inserted in a uniform static-guide magnetic field. Their studies are directed toward the design of a short-period wiggler to permit operation with low beam energy at short wavelengths. McDermott *et al.* [7] analyzed a wiggler consisting of a coaxial periodic permanent magnet and transmission line. This device was proposed for use in a high-power microwave source to drive a linear collider.

The present paper contains a study of the propagation of an electrostatic beam space-charge mode in a coaxial wiggler. The analysis is based on an idealized model of the wiggler magnetic field that neglects spatial harmonics of the wiggler wave number, radial variation of the magnetic-field magnitude, and waveguide boundary effects. In Sec. II, the motion of the electrons in the combined coaxial wiggler and axial-guide magnetic fields is analyzed. Equations for the electron velocity and the rate of change of the axial velocity component with the Lorentz factor are derived. In Sec. III, the dispersion relation for electrostatic beam space-charge waves is derived. In Sec. IV, the combined effects of the coaxial wiggler and axial-guide fields on the rate of change of axial velocity with energy and on the dispersion relation are studied numerically and some conclusions are presented.

II. QUASI-STEADY-STATE ANALYSIS

The static magnetic field in a coaxial magnetic wiggler will be represented by

$$B_0 = \hat{\mathbf{r}} B_{wr} \sin k_w r + \hat{\mathbf{z}}(B_{0g} + B_{wz} \cos k_w z). \tag{1}$$

Higher spatial harmonics of wiggler wave number k_w have been neglected. The radial variation of B_{wr} and B_{wz} will also be neglected, with the restriction that radial displacements of an electron be small in comparison to the wiggle period (i.e., $k_w|\delta r| \ll 1$). Note that a uniform axial-guide magnetic field B_{0g} is included. A cylindrical coordinate system and cgs Gaussian units will be employed. The electron velocity $\mathbf{v}_0$ satisfies the equation of motion

$$\frac{d\mathbf{v}_0}{dt} = \frac{-e}{\gamma_0 mc} \mathbf{v}_0 \times B_0, \tag{2}$$

where $-e$, m, and c are the electron charge, electron (rest) mass, and speed of light, respectively. Lorentz factor γ_0 is a constant given by

$$\gamma_0 = (1 - v_0^2/c^2)^{-1/2}, \tag{3}$$

where v_0 is the magnitude of the electron velocity

$$\mathbf{v}_0 = \hat{\mathbf{r}} v_{0r} + \hat{\boldsymbol{\theta}} v_{0\theta} + \hat{\mathbf{z}} v_{0z}. \tag{4}$$

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The equations of motion for the velocity components are

$$\frac{dv_{0r}}{dt} = -(\Omega_0 + \Omega_{wz} \cos k_w z) v_{0\theta} + r^{-1} v_{0\theta}^2, \tag{5}$$

$$\frac{dv_{0o}}{dt} = (\Omega_0 + \Omega_{wz} \cos k_w z) v_{0r} - (\Omega_{wr} \sin k_w z) v_{0z} \\ -r^{-1} v_{0r} v_{0\theta}, \tag{6}$$

$$\frac{dv_{0c}}{dt} = (\Omega_{wr} \sin k_w z) v_{0\theta}. \tag{7}$$

Here Ω_0, Ω_{wr}, and Ω_{wz} are the relativistic cyclotron frequencies corresponding to the axial-guide field B_{0g}, the radial wiggler field amplitude B_{wr}, and the axial wiggler field amplitude B_{wz}, respectively, given by

$$\Omega_0 = \frac{eB_{0g}}{\gamma_0 mc}, \tag{8}$$

$$\Omega_{wr} = \frac{eB_{wr}}{\gamma_0 mc}, \tag{9}$$

$$\Omega_{wz} = \frac{eB_{wz}}{\gamma_0 mc}. \tag{10}$$

Initial conditions will be chosen so that, in the limit of zero wiggler field, there is axial motion at constant velocity $v_\parallel$ but no Larmor motion. To first order in the wiggler field amplitude, Eqs. (5)–(7) lead to

$$\frac{d^2 v_{0r}}{dt^2} + \Omega_0^2 v_{0r} = \Omega_0 \Omega_{wr} v_\parallel \sin k_w z, \tag{11}$$

$$\frac{d^2 v_{0\theta}}{dt^2} + \Omega_0^2 v_{0\theta} = -\Omega_{wr} v_\parallel^2 k_w \cos k_w z, \tag{12}$$

and $z = v_\parallel t$. Solving these equations yields a quasi-steady-state solution for the electron velocity of the form

$$v_0 = \hat{\mathbf{r}} \Omega_0 k_w^{-1} \alpha \sin k_w z - \hat{\theta} v_\parallel \alpha \cos k_w z + \hat{\mathbf{z}} v_\parallel, \tag{13}$$

where

$$\alpha = \frac{\Omega_{wr} k_w v_\parallel}{\Omega_0^2 - k_w^2 v_\parallel^2}. \tag{14}$$

(The second-order correction to the axial velocity component is $\frac{1}{4} \alpha \Omega_{wr} k_w^{-1} \cos 2k_w z$.). The root-mean-square value of v_0 may be substituted into Eq. (3) to obtain

$$\frac{v_\parallel^2}{c^2} \left[1 + \frac{\Omega_{wr}^2 (\Omega_0^2 + k_w^2 v_\parallel^2)}{2(\Omega_0^2 - k_w^2 v_\parallel^2)^2} \right] = 1 - \gamma_0^{-2}. \tag{15}$$

This equation will be used to compute axial velocity $v_\parallel$ as a function of Ω_{wr}, Ω_0, k_w, and γ_0. Differentiating it with respect to γ_0 yields

$$\frac{dv_\parallel}{d\gamma_0} = \frac{c^2 \Phi_0}{\gamma_0 \gamma_\parallel^2 v_\parallel}, \tag{16}$$

where

$$\Phi_0 = 1 - \frac{\gamma_\parallel^2 \Omega_{wr}^2 \Omega_0^2 (\Omega_0^2 + 3k_w^2 v_\parallel^2)}{2(\Omega_0^2 - k_w^2 v_\parallel^2)^3 + \Omega_{wr}^2 \Omega_0^2 (\Omega_0^2 + 3k_w^2 v_\parallel^2)}. \tag{17}$$

A numerical study of Φ_0 will be made to determine the variation of the axial velocity with the electron energy.

III. ELECTROSTATIC WAVE ANALYSIS

The device under consideration consists of a coaxial wave guide containing a static magnetic field B_0. This field, which consists of both the wiggler field and the axial-guide field, is approximated by Eq. (1). In the unperturbed state, the electron density n_0 will be taken as uniform and time independent with the finite transverse dimensions of the beam ignored. The unperturbed electron fluid velocity v_0 will be taken as the first-order approximation given by Eq. (13). Effects of the unperturbed electric field $\mathbf{E}_0$ will be neglected.

In the presence of a small electrostatic perturbation, the total electric field **E**, magnetic field **B**, electron density n, and electron fluid velocity **v** may be expressed in the form

$$\mathbf{E} = \delta \mathbf{E}, \tag{18}$$

$$\mathbf{B} = \mathbf{B}_0, \tag{19}$$

$$n = n_0 + \delta n, \tag{20}$$

$$\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v}. \tag{21}$$

The small perturbation quantities satisfy Gauss's law

$$\nabla \cdot \delta \mathbf{E} = -4\pi e \delta n, \tag{22}$$

the linearized continuity equation

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta \mathbf{v} + \mathbf{v}_0 \cdot \delta n = 0, \tag{23}$$

and the linearized momentum-transfer equation

$$\frac{\partial \delta \mathbf{v}}{\partial t} + \mathbf{v}_0 \cdot \nabla \delta \mathbf{v} + \delta \mathbf{v} \cdot \nabla \mathbf{v}_0 \\ = -e(\gamma_0 m)^{-1} [\delta \mathbf{E} - c^{-2} \mathbf{v}_0 \mathbf{v}_0 \cdot \delta \mathbf{E} + c^{-1} \delta \mathbf{v} \times \mathbf{B}_0 \\ - \gamma_0^2 c^{-3} (\mathbf{v}_0 \times \mathbf{B}_0) \mathbf{v}_0 \cdot \delta \mathbf{v}]. \tag{24}$$

A small-amplitude plane wave with angular frequency w and wave number k that is propagating in the positive z direction will be assumed to comprise a perturbation of the form

$$\delta \mathbf{E} = \hat{\mathbf{z}} \delta \hat{E} \exp[i(kz - \omega t)], \tag{25}$$

$$\delta n = \delta \hat{n} \exp[i(kz - \omega t)], \tag{26}$$

$$\delta \mathbf{v} = (\hat{\mathbf{r}} \delta \hat{v}_r + \hat{\boldsymbol{\theta}} \delta \hat{v}_\theta + \hat{\mathbf{z}} \delta \hat{v}_z) \exp[i(kz - \omega t)]. \tag{27}$$

Here $\delta \hat{E}$, $\delta \hat{n}$, and $\delta \hat{v}_z$ are constants, and $\delta \hat{v}_r$ and $\delta \hat{v}_\theta$ are functions of z approximated by truncated Fourier series of the form

$$\delta \hat{v}_r = \delta \hat{v}_{r0} + \delta \hat{v}_{r1} \cos k_w z + \delta \hat{v}_{r2} \sin k_w z, \tag{28}$$

$$\delta\hat{v}_{\theta}=\delta\hat{v}_{\theta 0}+\delta\hat{v}_{\theta 1}\cos k_w z+\delta\hat{v}_{\theta 2}\sin k_w z. \tag{29}$$

Substitution of the assumed solution into Eqs. (22) and (23) yields

$$ik\delta\hat{E}=4\pi e\delta\hat{n}, \tag{30}$$

$$(\omega-kv_{\parallel})\delta\hat{n}=kn_0\delta\hat{v}_z. \tag{31}$$

Additional equations for the wave amplitudes may be obtained from Eq. (24) using the orthogonality of the Fourier-series function and neglecting higher spatial harmonics of k_w. This procedure leads to nine homogeneous algebraic equations in nine amplitudes ($\delta\hat{E}$, $\delta\hat{n}$, $\delta\hat{v}_z$, $\delta\hat{v}_{r0}$, $\delta\hat{v}_{r1}$, $\delta\hat{v}_{r2}$, $\delta\hat{v}_{\theta 0}$, $\delta\hat{v}_{\theta 1}$, and $\delta\hat{v}_{\theta 2}$). The necessary and sufficient condition for a nontrivial solution yields

$$(\omega-kv_{\parallel})^2=\frac{\omega_b^2\Phi}{\gamma_0\gamma_{\parallel}^2}, \tag{32}$$

where

$$\omega_b=(4\pi e^2 n_0/m)^{1/2} \tag{33}$$

is the nonrelativistic beam plasma frequency. Equation (32) is the dispersion relation for electrostatic beam space-charge waves. The function Φ deviates from unity due to the combined effects of the wiggler and axial magnetic fields. The condition for a nontrivial solution leads to a hierarchy of auxiliary algebraic equations, omitted herein for brevity, which will be used to compute Φ as a function of the system parameters.

IV. NUMERICAL STUDY AND CONCLUSIONS

Numerical calculations have been done to illustrate the combined effects of the coaxial wiggler and axial-guide fields on the variation of the electron axial velocity with electron energy and on the dispersion relation for beam space-charge waves. The amplitudes of the radial and axial components of the wiggler magnetic fields B_{wr} and B_{wz} were taken to be 1000 G and 100 G, respectively. Wiggler wavelength $2\pi/k_w$ and lab-frame electron density n_0 were taken to be 3 cm and 10^{12} cm^{-3}, respectively. Electron-beam energy $(\gamma_0-1)m_0c^2$ was taken to be 700 keV, corresponding to a Lorentz factor γ_0 of 2.37. The axial-guide magnetic field B_{0g} was varied from 0 to 16.9 kG, corresponding to a variation from 0 to 2 in the normalized relativistic cyclotron frequency Ω_0/ck_w associated with B_{0g}.

Figure 1 shows the variation of the axial velocity of the quasi-steady-state orbits with the axial guide magnetic field for two classes of solutions. Group I orbits (i.e., those for which $\Omega_0<k_wv_{\parallel}$) require for stability that the axial guide magnetic field not be too large ($\Omega_0/ck_w<0.69$). Unstable group I orbits are indicated by the dashed line. Group II orbits (i.e., those for which $\Omega_0>k_wv_{\parallel}$) are always stable; the axial velocity of the electrons increases from zero at zero guide field to highly relativistic velocities in a strong guide field. The curves shown in Fig. 1 for the coaxial wiggler are qualitatively the same as the corresponding curves for the helical wiggler and planar wiggler configurations (see Ref. [3]).

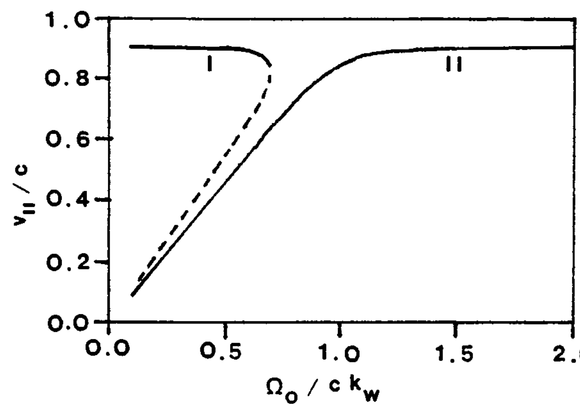

FIG. 1. Electron axial velocity component $v_{\parallel}$ (divided by the speed of light c) as a function of the normalized axial-guide magnetic field Ω_0/ck_w for group-I and group-II orbits.

The rate of change of the electron axial velocity with electron energy is given by Eq. (17). It is proportional to the function Φ_0 which is equal to unity in the absence of the wiggler field. Figure 2 illustrates the dependence of Φ_0 on the radial wiggler magnetic field and the axial guide magnetic field B_{0g}. For the group I orbits, Φ_0 is approximately equal to unity when the B_{0g} field is weak, but rises abruptly as B_{0g} increases to the value that results in orbital instability ($\Omega_0/ck_w\cong 0.69$). For group II orbits, Φ_0 is approximately equal to unity when B_{0g} is sufficiently large; it decreases with decreasing B_{0g}, becomes negative when B_{0g} is moderate ($\Omega_0/ck_w<1.1$), and approaches zero as B_{0g} approaches zero. The axial velocity decreases with increasing electron energy when Φ_0 is negative. Thus, a negative mass regime exists.

The dispersion relation [Eq. (32)] for electrostatic beam space-charge waves contains a factor Φ which multiplies the

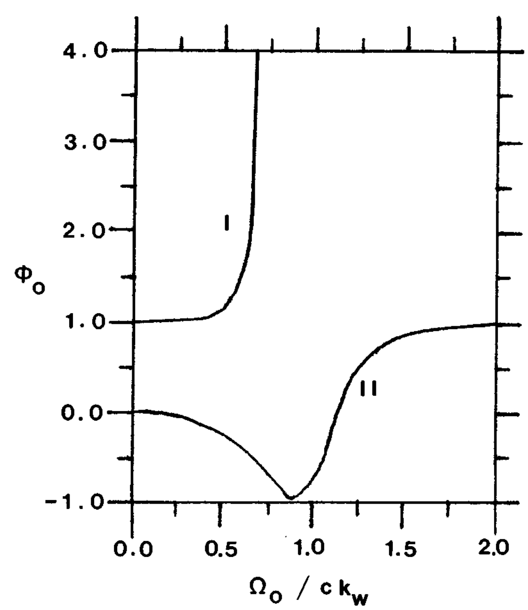

FIG. 2. Factor Φ_0 as a function of the normalized axial-guide magnetic field Ω_0/ck_w for group-I and group-II orbits.

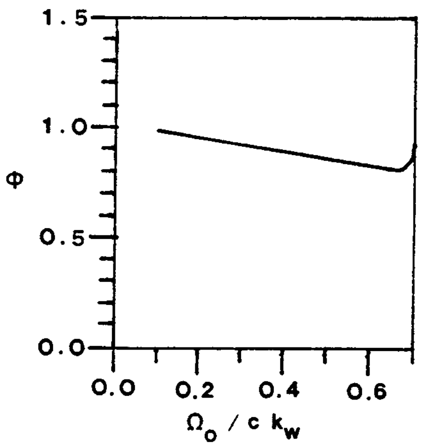

FIG. 3. Electron-density factor Φ as a function of the normalized axial-guide magnetic field Ω_0/ck_w for group-I orbits.

square of the plasma frequency. Consequently, the unperturbed electron density n_0 times Φ may be considered to be an effective electron density for these electrostatic waves. This electron-density factor deviates from unity due to the combined effects of the wiggler and guide fields. Figure 3 shows the variation of Φ with the axial-guide magnetic field B_{0g} for group-I orbits. This electron-density factor is somewhat below unity when B_{0g} is between zero and the value required for orbital instability. Figure 4 shows the variation of Φ with B_{0g} for group-II orbits. Note that Φ approaches unity as B_{0g} approaches zero or when it becomes sufficiently large. There are regions in which Φ is greater than unity as well as less than unity. Singular points exist at $\Omega_0/ck_w \cong 0.67$ and 1.19; at each of these points Φ fluctuates and does not approach a single value. Calculations made with $B_{wr}=B_{wz}=1000$ G yield results very similar to those shown in Fig. 4.

A physical interpretation of the modification of space-charge waves due to the combined wiggler and axial-guide magnetic fields may be described as follows. Electron oscillations along the beam axis are driven by the electric field.

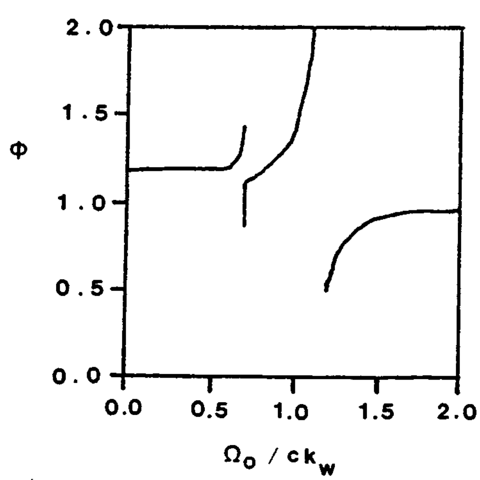

FIG. 4. Electron-density factor Φ as a function of the normalized axial-guide magnetic field Ω_0/ck_w for group-II orbits.

The magnetic $\mathbf{v}\times\mathbf{B}$ force due to these axial velocity fluctuations in the combined magnetic fields gives rise to velocity fluctuations in the radial and azimuthal directions. The resulting axial component of magnetic force modifies the axial motion. This is indicated by the axial momentum-transfer equation, which may be written in the form

$$-i(\omega-kv_{\parallel})\delta\hat{v}_z \exp[i(kz-\omega t)] = -e(\gamma_0 m)^{-1}[\delta\hat{E}-c^{-2}v_{\parallel}^2\delta\hat{E}+\hat{f}_z]\exp[i(kz-\omega t)]. \quad (34)$$

Here $\hat{f}_z$, which is the amplitude of the axial magnetic force per unit charge, is given by

$$\hat{f}_z=-(\gamma_0 me^{-1})\{\tfrac{1}{2}\Omega_{wr}[1-(\tfrac{1}{4})\eta]\delta\hat{v}_{\theta 2} + \tfrac{1}{8}\eta\Omega_{wr}\Omega_0(k_w v_{\parallel})^{-1}\delta\hat{v}_{r1}\}, \quad (35)$$

where

$$\eta=\gamma_0^2 v_{\parallel}^2 c^{-2}\alpha^2. \quad (36)$$

It may be expressed in the form

$$\hat{f}_z=\gamma_{\parallel}^{-2}(\Phi-1)\delta\hat{E}. \quad (37)$$

The amplitude of the total effective axial force per unit charge is

$$\delta\hat{E}-c^{-2}v_{\parallel}^2\delta\hat{E}+\hat{f}_z=\gamma_{\parallel}^{-2}[\delta\hat{E}+(\Phi-1)\delta\hat{E}]. \quad (38)$$

The axial momentum-transfer equation, Gauss's law, and the continuity equation then yield the dispersion relation [Eq. (32)].

When Φ is greater than unity, $\hat{f}_z$ is positive and this magnetic force acts in phase with the electric field, thereby increasing the frequency of the oscillations. The effective plasma frequency is then increased by the factor $\sqrt{\Phi}$. When Φ is less than unity, $\hat{f}_z$ is negative and opposes the direct action of the electric field, thereby decreasing the frequency of the oscillations. If Φ were to become negative, the effective plasma frequency would become purely imaginary as though the electrons have a negative mass. The resulting phase shift in axial motion would cause bunching to occur in such a way that the electric field would be enhanced and negative mass instability would result [1]. The present calculations show that, for a coaxial wiggler, a negative-mass regime exists. They also show that Φ can be less than unity, indicating that the magnetic force opposes the direct action of the electric field. This force does not become sufficiently large, however, to make the oscillations unstable.

In the analysis of the helical wiggler [1], two convenient approximations are employed, namely, the axial component of the wiggler field is neglected and the electron inertia is neglected in the transverse momentum-transfer equations. The latter approximation ensures that $\omega-kv_{\parallel}$ does not appear in the resulting equation for Φ. If these same two approximations are invoked for the coaxial wiggler, Φ is given by

$$\Phi \cong 1 - \frac{\gamma_\parallel^2 \Omega_{wr}^2 v_\parallel^2 c^{-2}[\Omega_0^2 + (k_w v_\parallel)^2]}{2[\Omega_0^2 - (k_w v_\parallel)^2]^2}. \qquad (39)$$

This indicates that, for group-II orbits, $\Phi \cong 1$ when Ω_0 is very large and decreases continuously as Ω_0 is decreased; e.g., $\Phi \cong 0.985$, 0.626, and 0.181 when $\Omega_0/ck_w = 2$, 1, and 0.1, respectively. Clearly, Φ does not become negative. Figures 3 and 4 show the Φ values when the inertia terms and axial wiggler field are restored. No negative mass instability is predicted.

The present analysis indicates that Φ_0 and Φ, which are approximately equal for helical and planar wigglers, are quite different for a coaxial wiggler. Furthermore, the helical and planar wigglers have negative-mass electrostatic instability regimes unlike the coaxial wiggler. This analysis is based on an idealized model. A more sophisticated theory could be developed, in principle, by including additional effects that arise from equilibrium electrostatic self-fields, wiggler magnetic-field harmonics, radial variations of the wiggler field, and the waveguide walls. Note that the coaxial wiggler proposed by McDermott *et al.* [7] does not contain an axial-guide magnetic field. It is anticipated that Raman free-electron lasers with a coaxial wiggler and an axial-guide field will ultimately be employed.

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